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## A block theoretic proof of Thompson's $A \times B$ -lemma

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**Abstract.** We show that Thompson's  $A \times B$ -lemma can be obtained as a consequence of Brauer's third main theorem.

**Mathematics Subject Classification.** 20D45, 20C05.

**Keywords.** Finite group, Automorphism, Group algebra, Block.

Let  $k$  be an algebraically closed field of prime characteristic  $p$ . Brauer's third main theorem [2, Theorem 3], rephrased using Brauer pairs (cf. [1, Theorem 3.13] or [7, Theorem 6.3.14], or also [5, Theorem 7] for a different proof), states that if  $b$  is the principal block idempotent of a finite group algebra  $kG$ , then  $\text{Br}_Q(b)$  is the principal block idempotent of  $kC_G(Q)$  for any  $p$ -subgroup  $Q$  of  $G$ . Here  $\text{Br}_Q : (kG)^Q \rightarrow kC_G(Q)$  denotes the Brauer homomorphism (cf. [3, §1.2] or [6, Theorem 5.4.1]). In particular, if  $kG$  has a unique block, then  $kC_G(Q)$  has a unique block. Moreover, if  $kG$  has a unique block, then  $O_{p'}(G) = 1$  because otherwise  $\frac{1}{|O_{p'}(G)|} \sum_{g \in O_{p'}(G)} g$  would be a central idempotent in  $kG$  different from 1. Also, it is well-known that if  $G$  has a self-centralising normal  $p$ -subgroup, then  $kG$  has a unique block (cf. [7, Corollary 6.2.8]). We use these facts to give a proof of the following result.

**Theorem 1** (Thompson's  $A \times B$ -lemma; cf. [4, Chapter 5, Theorem 3.4]). *Let  $A \times B$  be a subgroup of the automorphism group of a finite  $p$ -group  $P$ , with  $A$  a  $p'$ -group and  $B$  a  $p$ -group. If  $A$  acts trivially on  $C_P(B)$ , then  $A = 1$ .*

*Proof.* Consider the group  $G = P \rtimes (A \times B)$ , where the notation is as in the statement, and suppose that  $A$  acts trivially on  $C_P(B)$ . Note that  $S = P \rtimes B$  is the unique Sylow  $p$ -subgroup of  $G$ , and that  $A$  can be regarded as a  $p'$ -subgroup of the automorphism group of  $S$ . Thus  $S$  is self-centralising and normal in  $G$ , and hence  $kG$  has a unique block.

By the assumptions,  $A$  acts trivially on the group  $Q = C_P(B) \times B$ . That is, we have  $A \leq C_G(Q)$ , and hence  $C_G(Q) = C_S(Q) \rtimes A$ . Since  $kG$  has a unique block,  $kC_G(Q)$  has a unique block. We now show that  $Q$  is self-centralising in

$S$ . Let  $x \in C_S(Q)$ . Write  $x = yu$  for some  $y \in P$  and  $u \in B$ . Since  $u \in B$ , it follows that conjugation by  $u$  preserves the decomposition  $Q = C_P(B) \times B$ . Thus conjugation by  $y$  preserves this decomposition as well. In particular,  $y$  normalises  $B$ . By elementary group theory, it follows that  $y$  centralises  $B$ . Indeed, if  $u \in B$ , then  $yuy^{-1}u^{-1} \in P \cap B = 1$ . This shows that  $C_S(Q) \leq Q$ . Since  $Q \leq C_G(A)$ , it follows that  $C_G(Q) = C_S(Q) \times A$ . By the above,  $kC_G(Q)$  has a single block, and hence  $A = O_{p'}(C_G(Q)) = 1$ .  $\square$

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